

HIDDEN DYNAMICS AND THE ORIGIN OF PULSATING WAVES IN SELF-PROPAGATING HIGH TEMPERATURE SYNTHESIS

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ABSTRACT. We derive the precise limit of SHS in the high activation energy scaling suggested by B.J. Matkowsky-G.I. Sivashinsky in 1978 and by A. Bayliss-B.J. Matkowsky-A.P. Aldushin in 2002. In the time-increasing case the limit coincides with the Stefan problem for supercooled water *with spatially inhomogeneous coefficients*. In general it is a nonlinear forward-backward parabolic equation *with discontinuous hysteresis term*.

In the first part of our paper we give a complete characterization of the limit problem in the case of one space dimension.

In the second part we construct in any finite dimension a rather large family of pulsating waves for the limit problem.

In the third part, we prove that for constant coefficients the limit problem in any finite dimension *does not admit non-trivial pulsating waves*.

The combination of all three parts strongly suggests a relation between the pulsating waves constructed in the present paper and the numerically observed pulsating waves for finite activation energy in dimension $n \geq 1$ and therefore provides a possible and surprising explanation for the phenomena observed.

All techniques in the present paper (with the exception of the remark in the Appendix) belong to the category far-from-equilibrium-analysis/far-from-bifurcation-point-analysis.

1. INTRODUCTION

The system

$$(1) \quad \begin{aligned} \partial_t u - \Delta u &= v f(u) \\ \partial_t v &= -v f(u), \end{aligned}$$

where u is the normalized temperature, v is the normalized concentration of the reactant and the non-negative nonlinearity f describes the reaction kinetics, is a simple but widely used model for solid combustion (i.e. the case of the Lewis number being $+\infty$). In particular it is being used to model the industrial process of Self-propagating High temperature Synthesis (SHS). In the case of high activation energy interesting phenomena like instability of planar waves, fingering and screw-like spin combustion waves are observed.

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In [20] B.J. Matkowsky-G.I. Sivashinsky derived from a special case of (1) a formal singular limit containing a jump condition for the temperature on the interface. Later it has been argued that the problem is for high activation energy related to a Stefan problem describing the freezing of supercooled water (see [20], [9, p. 57]). Subsequently the Stefan problem for supercooled water became the basis for numerous papers focusing on stability analysis of (1), fingering, screw-like spin combustion waves etc. (see for example [9],[10],[12],[11],[13],[8],[2]).

Surprisingly there are few *mathematical* results on the subject: In [19] E. Logak-V. Loubeau proved existence of a planar wave in one space dimension and gave a rigorous proof for convergence as the activation energy goes to infinity.

Instability of the planar wave for a special linearization (and high activation energy) is due to [4].

The present paper consists of three parts: in the first part we prove rigorously that in the case of one space dimension the SHS system converges to the irreversible Stefan problem for supercooled water (cf. Theorem 5.1). In the time-increasing case we obtain also convergence in higher dimensions (see Theorem 4.1). As the initial data of the reactant concentration enter the equation as the activation energy goes to infinity, our result also seems to provide a *possible explanation for the numerically observed pulsating waves* (cf. [14], [22] and [2]).

As a matter of fact, in the second part of our paper (Theorem 7.1) we use the spatially inhomogeneous coefficients in order to construct a pulsating wave *for each periodic function* v^0 (or Y^0 , respectively) on \mathbf{R}^n , using the approach in [3]. We also obtain the spin combustion waves (or “helical waves”) on the cylinder mantle (see Remark 7.2).

In contrast, we show in the third part (see Theorem 8.1) that *for constant* v^0 in any finite dimension, *no non-trivial pulsating waves exist*. In addition, the formal result in the Appendix suggests that in one space dimension the planar wave is stable.

Taken together (cf. section 9), our results strongly suggest a relation between the pulsating waves constructed in the present paper and the numerically observed pulsating waves for finite activation energy in dimension $n \geq 1$ (cf. [14], [22] and [2]) and therefore provide a possible and surprising explanation for the phenomena observed.

In the original setting by B.J. Matkowsky-G.I. Sivashinsky [20, equation (2)], according to our result Theorem 5.1,

$$(2) \quad \begin{aligned} \partial_t u_N - \Delta u_N &= (1 - \sigma_N) N e^N v_N \exp(-N/u_N), \\ \partial_t v_N &= -N e^N v_N \exp(-N/u_N), \end{aligned}$$

each limit u_∞ of $u_N > 0$ as $N \rightarrow \infty$ satisfies for $(\sigma_N)_{N \in \mathbf{N}} \subset \subset [0, 1)$ (for $\sigma_N \uparrow 1, N \uparrow \infty$ the limit in this scaling is the solution of the heat equation; cf. Section 6.1 and Theorem 5.1)

$$(3) \quad \partial_t u_\infty - v^0 \partial_t \chi = \Delta u_\infty \text{ in } (0, +\infty) \times \Omega,$$

where v^0 are the initial data of v_∞ and

$$\chi(t, x) \begin{cases} = 0, & \text{esssup}_{(0,t)} u_\infty(\cdot, x) < 1, \\ \in [0, 1], & \text{esssup}_{(0,t)} u_\infty(\cdot, x) = 1, \\ = 1, & \text{esssup}_{(0,t)} u_\infty(\cdot, x) > 1. \end{cases}$$

In the SHS system with another scaling and a temperature threshold (see [2, p. 109-110]),

$$(4) \quad \begin{aligned} \partial_t \theta_N - \Delta \theta_N &= (1 - \sigma_N) N Y_N \exp((N(1 - \sigma_N)(\theta_N - 1))/(\sigma_N + (1 - \sigma_N)\theta_N)) \chi_{\{\theta_N > \bar{\theta}\}}, \\ \partial_t Y_N &= -(1 - \sigma_N) N Y_N \exp((N(1 - \sigma_N)(\theta_N - 1))/(\sigma_N + (1 - \sigma_N)\theta_N)) \chi_{\{\theta_N > \bar{\theta}\}} \end{aligned}$$

where $N(1 - \sigma_N) \gg 1$, $\sigma_N \in (0, 1)$ and $\bar{\theta} \in (0, 1)$, each limit θ_∞ of θ_N satisfies (cf. Section 6.2 and Theorem 5.1)

$$(5) \quad \partial_t \theta_\infty - Y^0 \partial_t \chi = \Delta \theta_\infty \text{ in } (0, +\infty) \times \Omega,$$

where Y^0 are the initial data of Y_∞ and

$$\chi(t, x) \begin{cases} = 0, & \text{esssup}_{(0,t)} \theta_\infty(\cdot, x) < 1, \\ \in [0, 1], & \text{esssup}_{(0,t)} \theta_\infty(\cdot, x) = 1, \\ = 1, & \text{esssup}_{(0,t)} \theta_\infty(\cdot, x) > 1. \end{cases}$$

To our knowledge this precise form of the limit problem, i.e. the equation with the discontinuous hysteresis term, has not been known. Even in the time-increasing case it does *not* coincide with the formal result in [20].

In the case that θ_∞ (or u_∞ , respectively) is increasing in time and v^0 (or Y^0 , respectively) is constant, our limit problem coincides with the Stefan problem for supercooled water, an extensively studied ill-posed problem (for a survey see [5]). As it is a forward-backward parabolic equation it is not clear whether one should expect uniqueness (see [6, Remark 7.2] for an example of non-uniqueness in a related problem).

On the positive side, much more is known about the Stefan problem for supercooled water than the SHS system, e.g. existence of a finger ([15]), instability of the finger ([18]), one-phase solutions ([6]) etc.; those results, when combined with our convergence result, suggest that similar properties should be true for the SHS system.

It is interesting to observe that even in the time-increasing case our singular limit *selects certain solutions* of the Stefan problem for supercooled water. For example, $u(t) = (\kappa - 1)\chi_{\{t < 1\}} + \kappa\chi_{\{t > 1\}}$ is for each $\kappa \in (0, 1)$ a perfectly valid solution of the Stefan problem for supercooled water, but, as easily verified, it cannot be obtained from the ODE

$$\partial_t u_\varepsilon(t) = -\partial_t \exp(-\frac{1}{\varepsilon} \int_0^t \exp((1 - 1/(u_\varepsilon(s) + 1))/\varepsilon) ds) \text{ as } \varepsilon \rightarrow 0.$$

Our approach does not involve stability or bifurcation analysis. For showing the convergence we use standard compactness and topological arguments. We prove that if the measure of the “burnt zone” is small enough in a parabolic cylinder, then in a cylinder of smaller radius there cannot be any burnt part. For the construction of pulsating waves we use the approach in [3] as well as blow-up arguments, Harnack inequality and so on. In order to obtain non-existence of pulsating waves for constant v^0 we use a Liouville technique.

Let us conclude the introduction with a comparison to blow-up in semilinear heat equations, as the main problem arising in our convergence proof, i.e. excluding “peaking of the solution” or burnt zones with very small measure, resembles the blow-up phenomena in semilinear heat equations. One could therefore hope to apply methods used to exclude blow-up in low dimensions in order to exclude peaking,

say in two dimensions. There are however problems: First, here, we are dealing not with a single solution but with the one-parameter family u_ε concentrating at some “peak” as ε gets smaller. Second, the ε -problem is not a scalar equation but a degenerate system. Third, in contrast to blow-up, peaking would not necessarily imply u_ε going to $+\infty$. Fourth, our limit problem is a two-phase problem while most known results for blow-up in semilinear heat equations assume the solution to be non-negative. Fifth, in our problem it does not make much sense studying the onset of burning, say the first time when $u_\varepsilon \geq -\varepsilon$, whereas studying the time of first blow-up can be very reasonable for semilinear heat equations. The last and most important difference is that while semilinear heat equations are parabolic and therefore well-posed in a sense, our limit problem contains a backward component making it *ill-posed*.

2. NOTATION

Throughout this article \mathbf{R}^n will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$. $B_r(x)$ will denote the open n -dimensional ball of center x , radius r and volume $r^n \omega_n$. When the center is not specified, it is assumed to be 0.

When considering a set A , χ_A shall stand for the characteristic function of A , while ν shall typically denote the outward normal to a given boundary. We will use the distance pardist with respect to the parabolic metric $d((t, x), (s, y)) = \sqrt{|t - s| + |x - y|^2}$.

The operator ∂_t will mean the partial derivative of a function in the time direction, Δ the Laplacian in the space variables and \mathcal{L}^n the n -dimensional Lebesgue measure. Finally $\mathbf{W}_p^{2,1}$ denotes the parabolic Sobolev space as defined in [17].

3. PRELIMINARIES

In what follows, Ω is a bounded C^1 -domain in \mathbf{R}^n and

$$u_\varepsilon \in \bigcap_{T \in (0, +\infty)} \mathbf{W}_2^{2,1}((0, T) \times \Omega)$$

is a strong solution of the equation

$$(6) \quad \begin{aligned} \partial_t u_\varepsilon(t, x) - \Delta u_\varepsilon(t, x) &= -v_\varepsilon^0(x) \partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_\varepsilon(u_\varepsilon(s, x)) ds\right), \\ u_\varepsilon(0, \cdot) &= u_\varepsilon^0 \text{ in } \Omega, \nabla u_\varepsilon \cdot \nu = 0 \text{ on } (0, +\infty) \times \partial\Omega; \end{aligned}$$

here g_ε is a non-negative function on \mathbf{R} satisfying:

0) g_ε is for each $\varepsilon \in (0, 1)$ piecewise continuous with only one possible jump at z_0 , $g_\varepsilon(z_0-) = g_\varepsilon(z_0) = 0$ in case of a jump, and g_ε satisfies for each $\varepsilon \in (0, 1)$ and for every $z \in \mathbf{R}$ the bound $g_\varepsilon(z) \leq C_\varepsilon(1 + |z|)$.

1) $g_\varepsilon/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ on each compact subset of $(-\infty, 0)$.

2) for each compact subset K of $(0, +\infty)$ there is $c_K > 0$ such that $\min(g_\varepsilon, c_K) \rightarrow c_K$ uniformly on K as $\varepsilon \rightarrow 0$.

The initial data satisfy $0 \leq v_\varepsilon^0 \leq C < +\infty$, v_ε^0 converges in $L^1(\Omega)$ to v^0 as $\varepsilon \rightarrow 0$, $(u_\varepsilon^0)_{\varepsilon \in (0, 1)}$ is bounded in $L^2(\Omega)$, it is uniformly bounded from below by a constant u_{\min} , and it converges in $L^1(\Omega)$ to u^0 as $\varepsilon \rightarrow 0$.

Remark 3.1. *Assumption 0) guarantees existence of a global strong solution for each $\varepsilon \in (0, 1)$.*

4. THE HIGH ACTIVATION ENERGY LIMIT

The following theorem has been proved in [21]. Let us repeat the statements and its proof for the sake of completeness.

Theorem 4.1. *The family $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is for each $T \in (0, +\infty)$ precompact in $L^1((0, T) \times \Omega)$, and each limit u of $(u_\varepsilon)_{\varepsilon \in (0,1)}$ as a sequence $\varepsilon_m \rightarrow 0$, satisfies in the sense of distributions the initial-boundary value problem*

$$(7) \quad \begin{aligned} \partial_t u - v^0 \partial_t \chi &= \Delta u \text{ in } (0, +\infty) \times \Omega, \\ u(0, \cdot) &= u^0 + v^0 H(u^0) \text{ in } \Omega, \quad \nabla u \cdot \nu = 0 \text{ on } (0, +\infty) \times \partial\Omega, \\ \text{where } \chi(t, x) &\begin{cases} \in [0, 1], & \text{esssup}_{(0,t)} u(\cdot, x) \leq 0, \\ = 1, & \text{esssup}_{(0,t)} u(\cdot, x) > 0, \end{cases} \end{aligned}$$

and H is the maximal monotone graph

$$H(z) \begin{cases} = 0, & z < 0, \\ \in [0, 1], & z = 0, \\ = 1, & z > 0. \end{cases}$$

Moreover, χ is increasing in time and u is a supercaloric function.

If $(u_\varepsilon)_{\varepsilon \in (0,1)}$ satisfies $\partial_t u_\varepsilon \geq 0$ in $(0, T) \times \Omega$, then u is a solution of the Stefan problem for supercooled water, i.e.

$$\partial_t u - v^0 \partial_t H(u) = \Delta u \text{ in } (0, +\infty) \times \Omega.$$

Remark 4.2. Note that assumption 1) is only needed to prove the second statement “If”.

Proof. Step 0 (Uniform Bound from below):

Since u_ε is supercaloric, it is bounded from below by the constant u_{\min} .

Step 1 ($L^2((0, T) \times \Omega)$ -Bound):

The time-integrated function $v_\varepsilon(t, x) := \int_0^t u_\varepsilon(s, x) ds$, satisfies

$$(8) \quad \partial_t v_\varepsilon(t, x) - \Delta v_\varepsilon(t, x) = w_\varepsilon(t, x) + u_\varepsilon^0(x)$$

where w_ε is a measurable function satisfying $0 \leq w_\varepsilon \leq C$. Consequently

$$\begin{aligned} \int_0^T \int_\Omega (\partial_t v_\varepsilon)^2 + \frac{1}{2} \int_\Omega |\nabla v_\varepsilon|^2(T) &= \int_0^T \int_\Omega (w_\varepsilon + u_\varepsilon^0) \partial_t v_\varepsilon \\ &\leq \frac{1}{2} \int_0^T \int_\Omega (\partial_t v_\varepsilon)^2 + \frac{T}{2} \int_\Omega (C + |u_\varepsilon^0|)^2, \end{aligned}$$

implying

$$(9) \quad \int_0^T \int_\Omega u_\varepsilon^2 \leq T \int_\Omega (C + |u_\varepsilon^0|)^2.$$

Step 2 ($L^2((0, T) \times \Omega)$ -Bound for $\nabla \min(u_\varepsilon, M)$):

For

$$G_M(z) := \begin{cases} z^2/2, & z < M, \\ Mz - M^2/2, & z \geq M, \end{cases}$$

and any $M \in \mathbf{N}$,

$$\int_\Omega G_M(u_\varepsilon) - G_M(u_\varepsilon^0) + \int_0^T \int_\Omega |\nabla \min(u_\varepsilon, M)|^2$$

$$= \int_0^T \int_{\Omega} -v_{\varepsilon}^0 \min(u_{\varepsilon}, M) \partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, x)) ds\right) ds.$$

As $\partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, x)) ds\right) \leq 0$, we know that $\partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, x)) ds\right)$ is bounded in $L^{\infty}(\Omega; L^1((0, T)))$, and

$$\begin{aligned} & \int_0^T \int_{\Omega} -v_{\varepsilon}^0 \min(u_{\varepsilon}, M) \partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, x)) ds\right) \\ & \leq C \int_{\Omega} \sup_{(0, T)} \max(\min(u_{\varepsilon}, M), 0) \leq CM \mathcal{L}^n(\Omega). \end{aligned}$$

Step 3 (Compactness):

Let $\chi_M : \mathbf{R} \rightarrow \mathbf{R}$ be a smooth non-increasing function satisfying $\chi_{(-\infty, M-1)} \leq \chi_M \leq \chi_{(-\infty, M)}$ and let Φ_M be the primitive such that $\Phi_M(z) = z$ for $z \leq M-1$ and $\Phi_M \leq M$. Moreover, let $(\phi_{\delta})_{\delta \in (0, 1)}$ be a family of mollifiers, i.e. $\phi_{\delta} \in C_0^{0,1}(\mathbf{R}^n; [0, +\infty))$ such that $\int \phi_{\delta} = 1$ and $\text{supp } \phi_{\delta} \subset B_{\delta}(0)$. Then, if we extend u_{ε} and v_{ε}^0 by the value 0 to the whole of $(0, +\infty) \times \mathbf{R}^n$, we obtain by the homogeneous Neumann data of u_{ε} that

$$\begin{aligned} & \partial_t (\Phi_M(u_{\varepsilon}) * \phi_{\delta})(t, x) \\ &= \left(\left(\chi_M(u_{\varepsilon}) \left(\chi_{\Omega} \Delta u_{\varepsilon} - v_{\varepsilon}^0 \partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, x)) ds\right) \right) \right) * \phi_{\delta} \right)(t, x) \\ &= \int_{\mathbf{R}^n} \chi_M(u_{\varepsilon})(t, y) \left(\chi_{\Omega}(y) \Delta u_{\varepsilon}(t, y) \right. \\ & \quad \left. - v_{\varepsilon}^0(y) \partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, y)) ds\right) \right) \phi_{\delta}(x - y) dy \\ &= \int_{\mathbf{R}^n} \phi_{\delta}(x - y) \left(-\chi'_M(u_{\varepsilon}(t, y)) \chi_{\Omega}(y) |\nabla u_{\varepsilon}(t, y)|^2 \right. \\ & \quad \left. - \chi_M(u_{\varepsilon}(t, y)) v_{\varepsilon}^0(y) \partial_t \exp\left(-\frac{1}{\varepsilon} \int_0^t g_{\varepsilon}(u_{\varepsilon}(s, y)) ds\right) \right. \\ & \quad \left. + \chi_M(u_{\varepsilon}(t, y)) \chi_{\Omega}(y) \nabla u_{\varepsilon}(t, y) \cdot \nabla \phi_{\delta}(x - y) \right) dy. \end{aligned}$$

Consequently

$$\int_0^T \int_{\mathbf{R}^n} |\partial_t (\Phi_M(u_{\varepsilon}) * \phi_{\delta})| \leq C_1(\Omega, C, M, \delta, T)$$

and

$$\int_0^T \int_{\mathbf{R}^n} |\nabla (\Phi_M(u_{\varepsilon}) * \phi_{\delta})| \leq C_2(\Omega, M, \delta, T).$$

It follows that $(\Phi_M(u_{\varepsilon}) * \phi_{\delta})_{\varepsilon \in (0, 1)}$ is for each (M, δ, T) precompact in $L^1((0, T) \times \mathbf{R}^n)$.

On the other hand

$$\begin{aligned} & \int_0^T \int_{\mathbf{R}^n} |\Phi_M(u_{\varepsilon}) * \phi_{\delta} - \Phi_M(u_{\varepsilon})| \leq C_3 \left(\delta^2 \int_0^T \int_{\Omega} |\nabla \Phi_M(u_{\varepsilon})|^2 \right)^{\frac{1}{2}} \\ & + 2(M - u_{\min}) T \mathcal{L}^n(B_{\delta}(\partial\Omega)) \leq C_4(C, \Omega, u_{\min}, M, T) \delta. \end{aligned}$$

Combining this estimate with the precompactness of $(\Phi_M(u_{\varepsilon}) * \phi_{\delta})_{\varepsilon \in (0, 1)}$ we obtain that $\Phi_M(u_{\varepsilon})$ is for each (M, T) precompact in $L^1((0, T) \times \mathbf{R}^n)$. Thus, by a diagonal sequence argument, we may take a sequence $\varepsilon_m \rightarrow 0$ such that $\Phi_M(u_{\varepsilon_m}) \rightarrow z_M$ a.e.

in $(0, +\infty) \times \mathbf{R}^n$ as $m \rightarrow \infty$, for every $M \in \mathbf{N}$. At a.e. point of the set $\{z_M < M - 1\}$, u_{ε_m} converges to z_M . At each point (t, x) of the remainder $\bigcap_{M \in \mathbf{N}} \{z_M \geq M - 1\}$, the value $u_{\varepsilon_m}(t, x)$ must for large m (depending on (M, t, x)) be larger than $M - 2$. But that means that on the set $\bigcap_{M \in \mathbf{N}} \{z_M \geq M - 1\}$, the sequence $(u_{\varepsilon_m})_{m \in \mathbf{N}}$ converges a.e. to $+\infty$. It follows that $(u_{\varepsilon_m})_{m \in \mathbf{N}}$ converges a.e. in $(0, +\infty) \times \Omega$ to a function $z : (0, +\infty) \times \Omega \rightarrow \mathbf{R} \cup \{+\infty\}$. But then, as $(u_{\varepsilon_m})_{m \in \mathbf{N}}$ is for each $T \in (0, +\infty)$ bounded in $L^2((0, T) \times \Omega)$, $(u_{\varepsilon_m})_{m \in \mathbf{N}}$ converges by Vitali's theorem (stating that a.e. convergence and a non-concentration condition in L^p imply in bounded domains L^p -convergence) for each $p \in [1, 2]$ in $L^p((0, T) \times \Omega)$ to the weak L^2 -limit u of $(u_{\varepsilon_m})_{m \in \mathbf{N}}$. It follows that $\mathcal{L}^{n+1}(\bigcap_{M \in \mathbf{N}} \{z_M \geq M - 1\}) = \mathcal{L}^{n+1}(\{u = +\infty\}) = 0$.

Step 4 (Identification of the Limit Equation in $\text{esssup}_{(0,t)} u > 0$):

Let us consider $(t, x) \in (0, +\infty) \times \Omega$ such that $u_{\varepsilon_m}(s, x) \rightarrow u(s, x)$ for a.e. $s \in (0, t)$ and $u(\cdot, x) \in L^2((0, t))$. In the case $\text{esssup}_{(0,t)} u(\cdot, x) > 0$, we obtain by Egorov's theorem and assumption 2) that $\exp(-\frac{1}{\varepsilon_m} \int_0^t g_{\varepsilon_m}(u_{\varepsilon_m}(s, x)) ds) \rightarrow 0$ as $m \rightarrow \infty$.

Step 5 (The case $\partial_t u_{\varepsilon} \geq 0$):

Let (t, x) be such that $u_{\varepsilon_m}(t, x) \rightarrow u(t, x) = \lambda < 0$: Then by assumption 1),

$$\exp(-\frac{1}{\varepsilon_m} \int_0^t g_{\varepsilon_m}(u_{\varepsilon_m}(s, x)) ds) \geq \exp(-t \frac{\max_{[u_{\min}, \lambda/2]} g_{\varepsilon_m}}{\varepsilon_m}) \rightarrow 1 \text{ as } m \rightarrow \infty.$$

Remark 4.3. We also obtain a rigorous convergence result in the case of (higher dimensional) traveling waves with suitable conditions at infinity. In this case our $L^2(W^{1,2})$ -estimate (Step 2) implies a no-concentration property of the time-derivative.

5. COMPLETE CHARACTERIZATION OF THE LIMIT EQUATION IN THE CASE OF ONE SPACE DIMENSION

The aim of this main section is the following theorem:

Theorem 5.1. Suppose in addition to the assumptions at the beginning of Section 4 that the space dimension $n = 1$ and that the initial data u_{ε}^0 converge in C^1 to a function u^0 satisfying $\nabla u^0 \neq 0$ on $\{u^0 = 0\}$. Then the family $(u_{\varepsilon})_{\varepsilon \in (0,1)}$ is for each $T \in (0, +\infty)$ precompact in $L^1((0, T) \times \Omega)$, and each limit u of $(u_{\varepsilon})_{\varepsilon \in (0,1)}$ as a sequence $\varepsilon_m \rightarrow 0$, satisfies in the sense of distributions the initial-boundary value problem

$$(10) \quad \begin{aligned} \partial_t u - v^0 \partial_t \chi &= \Delta u \text{ in } (0, +\infty) \times \Omega, \\ u(0, \cdot) &= u^0 + v^0 H(u^0) \text{ in } \Omega, \quad \nabla u \cdot \nu = 0 \text{ on } (0, +\infty) \times \partial\Omega, \end{aligned}$$

where H is the maximal monotone graph

$$H(z) \begin{cases} = 0, & z < 0, \\ \in [0, 1], & z = 0, \\ = 1, & z > 0 \end{cases}$$

$$\text{and } \chi(t, x) = H(\text{esssup}_{(0,t)} u(\cdot, x)) \begin{cases} = 0, & \text{esssup}_{(0,t)} u(\cdot, x) < 0, \\ \in [0, 1], & \text{esssup}_{(0,t)} u(\cdot, x) = 0, \\ = 1, & \text{esssup}_{(0,t)} u(\cdot, x) > 0, \end{cases}$$

Although we assume the space dimension from now on to be 1, we keep the multi-dimensional notation for the sake of convenience. Moreover we extend u_{ε} by even reflection at the lateral boundary to a space-periodic solution on $[0, +\infty) \times \mathbf{R}$.

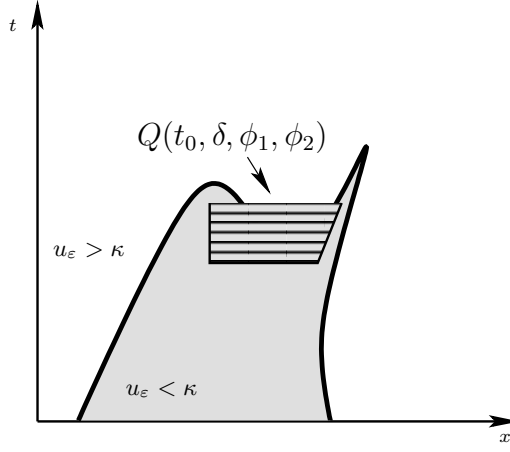


FIGURE 1. Clearing out

We start out with some elementary lemmata:

Lemma 5.2 (Clearing out). *There exists a continuous increasing function $\omega : [0, 1) \rightarrow [0, +\infty)$ such that $\omega(0) = 0$ and the following holds: suppose that $\kappa < 0$, that $\varepsilon \leq \omega(\kappa)$, that $\delta \in (0, 1)$ and that $u_\varepsilon \leq (1 + \omega(\delta))\kappa$ on the parabolic boundary of the domain $Q(t_0, \delta, \phi_1, \phi_2) := \{(t, x) : 0 \leq t_0 - 2\delta < t < t_0, \phi_1(t) < x < \phi_2(t)\}$, where $\phi_1 < \phi_2$ are C^1 -functions. Then $u_\varepsilon \leq \kappa$ in $Q(t_0, \delta, \phi_1, \phi_2)$ (cf. Figure 1).*

Proof. Comparing u_ε in $Q(t_0, \delta, \phi_1, \phi_2)$ to the solution of the ODE

$$y'(t) = Cg_\varepsilon(y)/\varepsilon, y(t_0 - 2\delta) = (1 + \omega(\delta))\kappa$$

we obtain the statement of the lemma.

Lemma 5.3. *For almost all $\kappa < 0$ the level set $((0, +\infty) \times \Omega) \cap \{u_\varepsilon = \kappa\}$ is a locally finite union of C^1 -curves. For such κ we define the set*

$$S_{\kappa, \varepsilon} := \{(t, x) \in (0, +\infty) \times \Omega) : u_\varepsilon(t, x) > \kappa$$

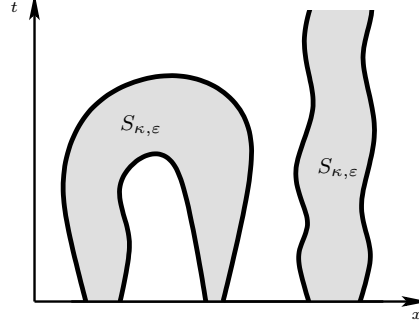
$$\text{and there is no } (t_0, \delta, \phi_1, \phi_2) \in [0, +\infty) \times (0, 1) \times C^1 \times C^1$$

$$\text{such that } u_\varepsilon \leq \kappa \text{ on the parabolic boundary of the domain } Q(t_0, \delta, \phi_1, \phi_2)\}$$

(cf. Figure 2). Then $\partial S_{\kappa, \varepsilon} = \bigcup_{j=1}^{N_{\kappa, \varepsilon}} \text{graph}(g_{j, \kappa, \varepsilon})$ where $g_{j, \kappa, \varepsilon} : [0, T_{j, \kappa, \varepsilon}] \rightarrow \mathbf{R}$ are piecewise C^1 -functions and $N_{\kappa, \varepsilon}$ is for small ε bounded by a constant depending only on the limit u^0 of the initial data.

Remark 5.4. *For illustration of the definition of $S_{\kappa, \varepsilon}$, imagine the set $\{u_\varepsilon > \kappa\}$ filled with water in a (t, x) -plane where t represents the height. Our modification of $\{u_\varepsilon > \kappa\}$ means then that the water is now allowed to flow out through the “bottom” $\{t = 0\}$.*

Proof of Lemma 5.3: By the definition of $S_{\kappa, \varepsilon}$ and by the fact that u_ε is supercaloric, each connected component of $\partial S_{\kappa, \varepsilon}$ is a piecewise C^1 -curve and touches $\{t = 0\}$. Therefore the number of connected components is for small $\varepsilon > 0$ bounded

FIGURE 2. The set $S_{\kappa, \varepsilon}$

by a constant \tilde{N} depending only on the limit u^0 of the initial data.

Let us consider one connected component γ of $\partial S_{\kappa, \varepsilon}$. By the definition of $S_{\kappa, \varepsilon}$ and by the fact that u_ε is supercaloric, the derivative of the time-component of the piecewise C^1 -curve γ can change its sign at most once! Thus we can define for each curve γ one or two piecewise C^1 -functions of time such that γ is the union of the graphs of the two functions. The total number of graphs $N_{\kappa, \varepsilon}$ is therefore bounded by $2\tilde{N}$.

Proof of Theorem 5.1:

By Theorem 4.1 we only have to prove that $\chi = 0$ in the set $\{\text{esssup}_{(0, t)} u(\cdot, x) < 0\}$. The main problem is to exclude “peaking” of the solution u_ε , i.e. tiny sets where $u_\varepsilon > \kappa$. Here we show that in the case of one space dimension, “peaking” is not possible. More precisely, if the measure of the set $u_\varepsilon > \kappa$ is small in a parabolic cube, then u_ε is strictly negative in the cube of half the radius, uniformly in ε . The proof is carried out in two steps:

Step 1: Let $(\varepsilon_m)_{m \in \mathbf{N}}$ be the subsequence in the proof of Theorem 4.1. As a.e. point $(t, x) \in ((0, +\infty) \times \mathbf{R}) \cap \{u < 0\}$ is a Lebesgue point of the set $\{u < 0\}$, we may assume that there exists $\kappa < 0$ such that for any $\theta \in (0, 1)$, sufficiently small $r_0 > 0$ and every $\varepsilon_m \in (0, \varepsilon_0)$,

$$\mathcal{L}^2(((t - 2r_0, t) \times B_{2r_0}(x)) \cap \{u_{\varepsilon_m} < 2\kappa\}) \geq \theta \mathcal{L}^2(((t - 2r_0, t) \times B_{2r_0}(x))) .$$

Step 2: Suppose now that $((t - r_0, t) \times B_{r_0}(x)) \cap \{u_{\varepsilon_m} > \kappa\} \neq \emptyset$ (where κ is chosen such that $\{u_{\varepsilon_m} = \kappa\}$ and $\{u_{\varepsilon_m} = 2\kappa\}$ are locally finite unions of C^1 -curves): then $((t - r_0, t) \times B_{r_0}(x)) \cap \partial S_{\kappa, \varepsilon_m}$ and $((t - r_0, t) \times B_{r_0}(x)) \cap \partial S_{2\kappa, \varepsilon_m}$ must by Lemma 5.3 be connected to the parabolic boundary of $(t - 2r_0, t) \times B_{2r_0}(x)$. The $L^2((0, T) \times \Omega)$ -Bound for $\nabla \min(u_\varepsilon, M)$, the fact that $\mathcal{L}^2(((t - 2r_0, t) \times B_{2r_0}(x)) \cap \{u_{\varepsilon_m} < 2\kappa\}) \geq \theta \mathcal{L}^2(((t - 2r_0, t) \times B_{2r_0}(x)))$ and Lemma 5.2 imply now (see Figure 3) that there must be an “almost horizontal” component of $\partial S_{\kappa, \varepsilon_m}$ (cf. Figure 4) with the following properties:

for any $\delta \in (0, 1)$, there are $t - r_0 < t_1 < t_2 < t_3 < t$ such that (see Figure 5)

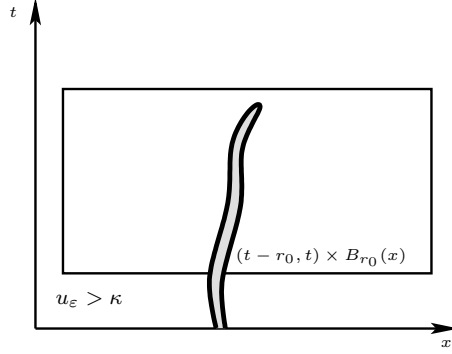
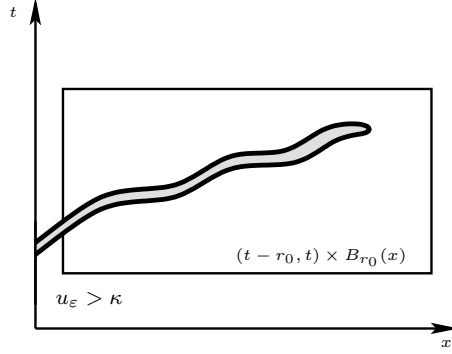
FIGURE 3. Situation excluded by the $L^2(W^{1,2})$ -estimate

FIGURE 4. The main task is to exclude almost horizontal propagation

$t_3 - t_1 \rightarrow 0$ as $\varepsilon_m \rightarrow 0$, for some j

$$|g_{j,\kappa,\varepsilon_m}(t_2) - g_{j,\kappa,\varepsilon_m}(t_1)| \geq c_1 > 0,$$

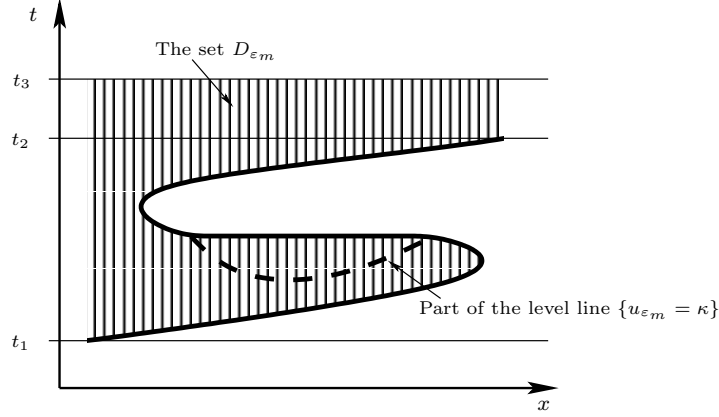
and

$$\begin{aligned} \mathcal{L}^1(\{y \in B_{r_0}(x) : u_{\varepsilon_m}(t_3, y) > 2\kappa\}) &\leq \delta, \\ \int_{B_{r_0}(x) \cap \{u_{\varepsilon_m}(t_3, \cdot) > 2\kappa\}} |u_{\varepsilon_m}(t_3, y)| dy &\leq \delta. \end{aligned}$$

We may assume that $c_1 < r_0$, that $g_{j,\kappa,\varepsilon_m}(t_2) = \sup_{(t_1, t_2)} g_{j,\kappa,\varepsilon_m}$, that $g_{j,\kappa,\varepsilon_m}(t_2) > g_{j,\kappa,\varepsilon_m}(t_1)$ and that $u_{\varepsilon_m}(s, y) > \kappa$ for some $d > 0$ and $(s, y) \in (t_1, t_2) \times B_{r_0}(x)$ such that $g_{j,\kappa,\varepsilon_m}(s) < y < d + g_{j,\kappa,\varepsilon_m}(s)$. We define the set $D_{\varepsilon_m} :=$

$\{(s, y) : t_1 < s < t_3, y < g_{j,\kappa,\varepsilon_m}(s) \text{ for } s \in (t_1, t_2) \text{ and } y < g_{j,\kappa,\varepsilon_m}(t_2) \text{ for } s \in [t_2, t_3]\}$
(cf. Figure 5) and the cut-off function $\phi(y) := \max(0, \min(y - g_{j,\kappa,\varepsilon_m}(t_1), g_{j,\kappa,\varepsilon_m}(t_2) - y))$. It follows that

$$c_1^2 \kappa / 4 + 2\delta + o(1) \geq o(1) + \int_{g_{j,\kappa,\varepsilon_m}(t_1)}^{g_{j,\kappa,\varepsilon_m}(t_2)} \phi(y) (u_{\varepsilon_m}(t_3, y) - \kappa) dy$$

FIGURE 5. The set D_{ε_m}

$$\begin{aligned}
&\geq o(1) + \int_{g_{j,\kappa,\varepsilon_m}(t_1)}^{g_{j,\kappa,\varepsilon_m}(t_2)} \phi(y) u_{\varepsilon_m}(t_3, y) dy - \kappa \int_{t_1}^{t_2} \phi(g_{j,\kappa,\varepsilon_m}(s)) g'_{j,\kappa,\varepsilon_m}(s) ds \\
&\geq \int_{g_{j,\kappa,\varepsilon_m}(t_1)}^{g_{j,\kappa,\varepsilon_m}(t_2)} \phi(y) u_{\varepsilon_m}(t_3, y) dy - \int_{t_1}^{t_2} \phi(g_{j,\kappa,\varepsilon_m}(s)) u_{\varepsilon_m}(s, g_{j,\kappa,\varepsilon_m}(s)) g'_{j,\kappa,\varepsilon_m}(s) ds \\
&\geq \int_{D_{\varepsilon_m}} \phi \partial_t u_{\varepsilon_m} \geq \int_{D_{\varepsilon_m}} \phi \Delta u_{\varepsilon_m} \\
&= - \int_{D_{\varepsilon_m}} \nabla \phi \cdot \nabla u_{\varepsilon_m} + \int_{t_1}^{t_2} \phi(s, g_{j,\kappa,\varepsilon_m}(s)) \partial_x u_{\varepsilon_m}(s, g_{j,\kappa,\varepsilon_m}(s)) \chi_{\{u_{\varepsilon_m}(s, g_{j,\kappa,\varepsilon_m}(s)) = \kappa\}} ds \\
&\geq - \int_{D_{\varepsilon_m}} \nabla \phi \cdot \nabla u_{\varepsilon_m} \rightarrow 0 \text{ as } \varepsilon_m \rightarrow 0,
\end{aligned}$$

a contradiction for small ε_m provided that δ has been chosen small enough; in the third inequality we used Lemma 5.2, and the convergence to 0 is due to the uniform $L^2(W^{1,2})$ -bound.

6. APPLICATIONS

Although the limit equation is an ill-posed problem, the convergence to the limit seems to be robust with respect to perturbations of the ε -system and the scaling: here we mention two examples of different systems leading to the same limit. Other examples can be found in mathematical biology (see [16] and [24]).

For the convergence results below we assume that the space dimension is 1.

6.1. The Matkowsky-Sivashinsky scaling. We apply our result to the scaling in [20, equation (2)], i.e.

$$(11) \quad \begin{aligned} \partial_t u_N - \Delta u_N &= (1 - \sigma_N) N v_N \exp(N(1 - 1/u_N)), \\ \partial_t v_N &= -N v_N \exp(N(1 - 1/u_N)), \end{aligned}$$

where the normalized temperature u_N and the normalized concentration v_N are non-negative, $(\sigma_N)_{N \in \mathbf{N}} \subset \subset [0, 1]$ (in the case $\sigma_N \uparrow 1, N \uparrow \infty$ the limit equation in the scaling as it would be the heat equation, but we could still apply our result to $u_N/(1 - \sigma_N)$) and the activation energy $N \rightarrow \infty$.

Setting $u_{\min} := -1, \varepsilon := 1/N, u_\varepsilon := u_N - 1$ and

$$g_\varepsilon(z) := \begin{cases} \exp((1 - 1/(z + 1))/\varepsilon), & z > -1 \\ 0, & z \leq -1 \end{cases}$$

and integrating the equation for v_N in time, we see that the assumptions of Theorem 5.1 are satisfied and we obtain that each limit u_∞, σ_∞ of u_N, σ_N satisfies

$$(12) \quad \partial_t u_\infty - (1 - \sigma_\infty) v^0 \partial_t H(\text{esssup}_{(0,t)} u_\infty) = \Delta u_\infty \text{ in } (0, +\infty) \times \Omega,$$

$$u_\infty(0, \cdot) = u^0 + v^0 H(u^0) \text{ in } \Omega, \quad \nabla u_\infty \cdot \nu = 0 \text{ on } (0, +\infty) \times \partial\Omega,$$

where v^0 are the initial data of v_∞ . Moreover, χ is increasing in time and u_∞ is a supercaloric function.

6.2. SHS in another scaling with temperature threshold. Here we consider (cf. [2, p. 109-110]), i.e.

$$(13) \quad \begin{aligned} \partial_t \theta_N - \Delta \theta_N &= (1 - \sigma_N) N Y_N \exp((N(1 - \sigma_N)(\theta_N - 1))/(\sigma_N + (1 - \sigma_N)\theta_N)) \chi_{\{\theta_N > \bar{\theta}\}}, \\ \partial_t Y_N &= -(1 - \sigma_N) N Y_N \exp((N(1 - \sigma_N)(\theta_N - 1))/(\sigma_N + (1 - \sigma_N)\theta_N)) \chi_{\{\theta_N > \bar{\theta}\}} \end{aligned}$$

where $N(1 - \sigma_N) \gg 1, \sigma_N \in (0, 1)$ and the constant $\bar{\theta} \in (0, 1)$ is a threshold parameter at which the reaction sets in.

Setting $u_{\min} = -1, \varepsilon := 1/(N(1 - \sigma_N)), \kappa(\varepsilon) := 1 - \sigma_N, u_\varepsilon := \theta_N - 1,$

$$g_\varepsilon(z) := \begin{cases} \exp((z/(\kappa(\varepsilon)z + 1))/\varepsilon), & z > \bar{\theta} - 1 \\ 0, & z \leq \bar{\theta} - 1 \end{cases}$$

and integrating the equation for Y_N in time, we see that the assumptions of Theorem 5.1 are satisfied and we obtain that each limit u_∞ of u_N satisfies

$$(14) \quad \partial_t u_\infty - v^0 \partial_t H(\text{esssup}_{(0,t)} u_\infty) = \Delta u_\infty \text{ in } (0, +\infty) \times \Omega,$$

$$u_\infty(0, \cdot) = u^0 + v^0 H(u^0) \text{ in } \Omega, \quad \nabla u_\infty \cdot \nu = 0 \text{ on } (0, +\infty) \times \partial\Omega,$$

where v^0 are the initial data of v_∞ . Moreover, χ is increasing in time and u_∞ is a supercaloric function.

7. EXISTENCE OF PULSATING WAVES

The aim of this section is to construct pulsating waves for the limit problem. For the sake of clarity we have chosen not to present the most general result in the following theorem. Moreover we confine ourselves to the one-phase case.

Theorem 7.1. (Existence of pulsating waves)

Let us consider a Hölder continuous function v^0 defined on \mathbf{R}^n that satisfies

$$v^0(x) \geq 1 \quad \text{and} \quad v^0(x+k) = v^0(x) \quad \text{for every } k \in \mathbf{Z}^n, \quad x \in \mathbf{R}^n.$$

Given a unit vector $e \in \mathbf{R}^n$ and a velocity $c > 0$, there exists a solution $u(t, x)$ of the one-phase problem

$$(15) \quad \begin{cases} \partial_t u - v^0 \partial_t \chi_{\{u \geq 0\}} = \Delta u & \text{on } \mathbf{R} \times \mathbf{R}^n \\ \partial_t u \geq 0 \quad \text{and} \quad -\mu_0 := -\int_{[0,1]^n} v^0 \leq u \leq 0, \end{cases}$$

which satisfies

$$(16) \quad \begin{cases} u(t, x+k) = u(t - \frac{e \cdot k}{c}, x) & \text{for every } k \in \mathbf{Z}^n, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n \\ u(t, x) = 0 & \text{for } x \cdot e - ct \leq 0 \quad \text{and} \quad \limsup_{x \cdot e - ct \rightarrow +\infty} u(t, x) = -\mu_0, \end{cases}$$

where the last limit is uniform as $x \cdot e - ct$ tends to $+\infty$.

Remark 7.2. By modifications of the following proofs and of the theory in [3], it is possible to replace \mathbf{R}^n in Theorem 7.1 by a smooth source manifold. Taking for example $S^1 \times \mathbf{R}$, we obtain the screw-like pulsating waves observed in spin combustion (also called “helical waves”; see for example [14], [22], [2], [14]).

Let us transform the problem by the so-called Duvaut transform (see [23]), setting $w(t, x) = -\int_t^{+\infty} u(s, x) ds$. In this section we will prove the existence of a pulsating wave w . More precisely, Theorem 7.1 is a corollary of the following result which will be proved later.

Theorem 7.3. (Pulsating waves for the obstacle problem)

Under the assumptions of Theorem 7.1, there exists a function $w(t, x)$ solving the obstacle problem

$$(17) \quad \begin{cases} \partial_t w = \Delta w - v^0 \chi_{\{w > 0\}} & \text{on } \mathbf{R} \times \mathbf{R}^n, \\ w \geq 0, \quad -\mu_0 \leq \partial_t w \leq 0, \quad \partial_{tt} w \geq 0, \end{cases}$$

with the conditions

$$(18) \quad \begin{cases} w(t, x+k) = w(t - \frac{e \cdot k}{c}, x) & \text{for every } k \in \mathbf{Z}^n, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n, \\ w(t, x) = 0 & \text{for } x \cdot e - ct \leq 0 \quad \text{and} \quad \partial_t w(t, x) \longrightarrow -\mu_0 \text{ as } x \cdot e - ct \longrightarrow +\infty. \end{cases}$$

The convergence is uniform as $x \cdot e - ct$ tends to $+\infty$.

Proof of Theorem 7.1

Simply set $u(t, x) := \partial_t w(t, x)$ with w given by Theorem 7.3, and use the fact that $\chi_{\{u < 0\}} = \chi_{\{w > 0\}}$. To check this last property, it is sufficient to exclude the case where $w(t_0, x_0) > 0$ and $\partial_t w(t_0, x_0) = 0$ at some point (t_0, x_0) : using the fact that $\partial_t w$ is caloric in $\{w > 0\}$ as well as the strong maximum principle, we deduce that $\partial_t w(t, x) = 0$ for $t \in (-\infty, t_0]$ and x in a neighborhood of x_0 . This contradicts the last line of (18).

Proof of Theorem 7.3

We will prove the existence of an unbounded solution w in six steps, approximating w by bounded solutions of a truncated equation, for which we can apply the existence of pulsating fronts due to Berestycki and Hamel [3].

Step 1: Approximation by bounded solutions and estimates of the velocity

For any $0 < A < M$, let us start by approximating the function $\chi_{(0,+\infty)}$ by the characteristic function $g = \chi_{(0,A]}$. In that case we can compute explicitly the *traveling wave* (ϕ, c_0) (unique up to translations of ϕ) of

$$(19) \quad c_0 \phi' = \phi'' - g(\phi), \quad \phi' \leq 0 \quad \text{on } \mathbf{R}, \quad \phi(-\infty) = M \quad \text{and} \quad \phi(+\infty) = 0 \quad :$$

Let us define for $c_0 > 0, M > 0, s_0 \in (-\infty, 0)$ and $s_1 \in (s_0, 0)$

$$\phi(s) = \begin{cases} M(1 - e^{c_0(s-s_0)}) & \text{for } s \in (-\infty, s_1] , \\ \frac{1}{c_0^2}(e^{c_0 s} - 1 - c_0 s) & \text{for } s \in [s_1, 0] , \\ 0 & \text{for } s \in [0, +\infty) . \end{cases}$$

For any $A \in (0, M)$ and for suitable s_0, s_1 , we see that ϕ is continuous and satisfies $\phi(s_1) = A$, which fixes the parameter s_1 as a function of A . Moreover we see that ϕ is of class C^1 if and only if $s_1 = -c_0 M$ and

$$(20) \quad M - A = \frac{1}{c_0^2} (1 - e^{-c_0^2 M}) .$$

Thus A is determined in terms of the velocity c_0 and M . The above calculations show in particular that $\phi(c_0 t - e \cdot x)$ is a good bounded approximation of the solution of (17) -(18) in the case $v^0 = 1$, i.e. the traveling wave case.

In all that follows let A be given by (20).

Now, when the function g in (19) is replaced by a Lipschitz continuous function whose support is a compact interval, there are known results on the existence of pulsating waves. For such g , it is possible to apply Theorem 1.13 of Berestycki, Hamel [3], which states the existence (and uniqueness up to translation in time) of bounded pulsating solutions traveling at a unique velocity. Bearing that in mind, we define g_M as a Lipschitz regularization of the characteristic function g such that $\text{supp } g_M = [0, A]$ and – for later use –

$$(21) \quad \begin{aligned} g_M &= 1 \quad \text{on } [1/M, A/2], \quad 0 \leq g_M \leq 1 \quad \text{on } \mathbf{R}, \\ \text{and} \quad g'_M &\geq 0, \quad g''_M \leq 0 \quad \text{on } (0, A/2) . \end{aligned}$$

Let us call c_M^0 the unique velocity of the traveling wave equation (19) with g replaced by g_M . As (ϕ, c_0) can be shown to be unique up to translations of ϕ , $c_M^0 \rightarrow c_0$ as $g_M \rightarrow g$.

Then there exists by Theorem 1.13 of [3] a bounded pulsating wave w_M traveling at velocity c_M such that

$$\left\{ \begin{array}{l} \partial_t w_M = \Delta w_M - v^0 g_M(w_M) \quad \text{on } \mathbf{R} \times \mathbf{R}^n , \\ \partial_t w_M \leq 0 \quad \text{and} \quad \limsup_{x \cdot e - c_M t \rightarrow -\infty} w_M(t, x) = 0 \leq w_M \leq M = \liminf_{x \cdot e - c_M t \rightarrow +\infty} w_M(t, x) \\ \text{and} \quad w_M(t, x + k) = w_M(t - \frac{e \cdot k}{c_M}, x) \quad \text{for every } k \in \mathbf{Z}^n, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n . \end{array} \right.$$

From the assumption $1 \leq v^0$ and the comparison principle (see Lemma 3.2 and 3.4 of [3]) we infer that the velocities satisfy the ordering $c_M^0 \leq c_M$. Similarly, defining $\lambda = \|v^0\|_{L^\infty}$ and comparing to $w_M(\lambda t, \sqrt{\lambda}x)$, we get $c_M \leq c_M^0 \sqrt{\lambda}$. Furthermore the above comparison principles tell us that the velocity c_M (resp. c_M^0) is continuous and non-decreasing in M .

For all that follows let $c > 0$ be an arbitrary but fixed velocity for which we want to construct the pulsating wave. Then, for any $M > 0$ we can adjust $A \in (0, M)$ such that for c_0 defined by (20),

$$c = c_M \in [c_0/2, 2c_0\sqrt{\|v^0\|_{L^\infty}}] .$$

In order to pass to the limit as $M \rightarrow +\infty$, we need to get some bounds on the solution first. To this end, rotating space-time proves to be very convenient:

Step 2: Space-time transformation and first estimates on the time derivatives

Let us introduce the function \tilde{w}_M defined by $\tilde{w}_M(s, x) = w_M(\frac{s+e \cdot x}{c}, x)$ which is periodic in x and satisfies

$$L\tilde{w}_M = v^0(x)g_M(\tilde{w}_M) \quad \text{with} \quad L\tilde{w}_M = \Delta\tilde{w}_M + \partial_{ss}\tilde{w}_M - 2\partial_{e,s}\tilde{w}_M - c\partial_s\tilde{w}_M$$

and $\lim_{s \rightarrow -\infty} \tilde{w}_M(s, x) = M$, $\lim_{s \rightarrow +\infty} \tilde{w}_M(s, x) = 0$ uniformly with respect to x . Using (21), we obtain $L\partial_s\tilde{w}_M - c_1\partial_s\tilde{w}_M = 0$ and $L\partial_{ss}\tilde{w}_M - c_1\partial_{ss}\tilde{w}_M \leq 0$ on $\{\tilde{w}_M < A/2\}$ for $c_1(s, x) = v^0(x)g'_M(\tilde{w}_M(s, x)) \geq 0$ on this set. We deduce from the maximum principle (see Lemma 3.2 and 3.4 of [3]) that for any $s_0 \in \mathbf{R}$ such that $\sup_{[s_0, +\infty) \times \mathbf{R}^n} \tilde{w}_M \leq A/2$,

$$(22) \quad \begin{aligned} \inf_{[s_0, +\infty) \times \mathbf{R}^n} \partial_s \tilde{w}_M &= \inf_{\mathbf{R}^n} \partial_s \tilde{w}_M(s_0, \cdot) \\ \text{and} \quad \inf_{[s_0, +\infty) \times \mathbf{R}^n} \partial_{ss} \tilde{w}_M &= \min(0, \inf_{\mathbf{R}^n} \partial_{ss} \tilde{w}_M(s_0, \cdot)) . \end{aligned}$$

Step 3: Bound of the solution from above

From the fact that g_M is bounded by 1, and from the Harnack inequality, we deduce that there exists a constant $C_H \in (1, +\infty)$ such that for any $r > 0$ and for any point (t_0, x_0)

$$(23) \quad \sup_{B_r(x_0)} w_M(t_0 - r^2, \cdot) \leq C_H \left(\inf_{B_r(x_0)} w_M(t_0, \cdot) + r^2 \lambda \right)$$

where $\lambda = \|v^0\|_{L^\infty}$. For \tilde{w}_M that means that – setting $s_0 = ct_0 - e \cdot x_0 -$

$$\sup_{y \in B_{\sqrt{n}/2}(0)} \tilde{w}_M(s_0 - cr^2 - e \cdot y, x_0 + y) \leq C_H \left(\inf_{y \in B_{\sqrt{n}/2}(0)} \tilde{w}_M(s_0 - e \cdot y, x_0 + y) + r^2 \lambda \right)$$

for $r \geq \sqrt{n}/2$. We will now use the fact that the unit cell $(-1/2, 1/2)^n$ is contained in the ball $B_{\sqrt{n}/2}(0)$. Using first the monotonicity of \tilde{w}_M in the variable s , and second the periodicity of $\tilde{w}_M(\tau, y)$ in y , we get for $\tau_0 := s_0 - \sqrt{n}/2$

$$(24) \quad \sup_{\mathbf{R}^n} \tilde{w}_M(\tau_0 - cr^2 + \sqrt{n}, \cdot) \leq C_H \left(\inf_{\mathbf{R}^n} \tilde{w}_M(\tau_0, \cdot) + r^2 \lambda \right) .$$

By a translation in time we may assume that

$$(25) \quad 0 = \inf \{ \tau : \quad \tilde{w}_M(s, x) \leq 1/M \quad \text{for} \quad s \geq \tau, \quad x \in \mathbf{R}^n \}$$

and get the bound

$$(26) \quad \tilde{w}_M(s, x) \leq \max(1/M, \alpha - \beta s)$$

for some constants $\alpha, \beta \in (0, +\infty)$ and every large positive M .

Step 4: Passing to the limit

By estimate (26), we can pass to the limit as $M \rightarrow +\infty$ and obtain $M - A \rightarrow 1/c_0^2$. Moreover, passing to a subsequence if necessary, \tilde{w}_M converges locally in $\mathbf{W}_p^{2,1}$ to \tilde{w} satisfying

$$\begin{cases} \partial_t w \leq 0 & \text{and} & \limsup_{x \cdot e - ct \rightarrow -\infty} w(t, x) = 0 \leq w \\ \text{and} & w(t, x + k) = w(t - \frac{e \cdot k}{c}, x) & \text{for every } k \in \mathbf{Z}^n, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n. \end{cases}$$

Furthermore, we obtain for w related to \tilde{w} by $\tilde{w}(s, x) = w(\frac{s+e \cdot x}{c}, x)$ that

$$w_t = \Delta w - v^0 \chi_{\{w > 0\}};$$

here we used the fact that w , being locally a $\mathbf{W}_p^{2,1}$ -function, satisfies $\partial_t w = 0 = \Delta w$ a.e. on the set $\{w = 0\}$.

In order to conclude $\partial_{tt} w \geq 0$ the following non-degeneracy property will prove to be necessary:

Step 5: Non-degeneracy property and bound from below

Let us assume that $w_M(t_0, x_0) \in (1/M, A/2)$. Using the fact that $v^0(x)g_M(z) \geq 1$ for $z \in [1/M, A/2]$, we can use the usual parabolic maximum principle, comparing $\max(w_M, 1/M)$ to the function

$$h(t, x) = w_M(t_0, x_0) + \frac{1}{4n}|x - x_0|^2 + \frac{1}{4n}(t_0 - t)$$

on the set

$$\{1/M < w_M < A/2\} \cap Q_r^-(t_0, x_0),$$

where $Q_r^-(t_0, x_0) = \{(t, x) : t_0 - r^2 \leq t \leq t_0, |x - x_0| \leq r\}$. We get for every $r > 0$ the following non-degeneracy property:

$$(27) \quad \sup_{Q_r^-(t_0, x_0)} w_M \geq \min\left(w_M(t_0, x_0) + \frac{1}{4n}r^2, A/2\right)$$

Combined with the Harnack-type inequality (24) for some radius $r' < r$, we obtain the following bound from below:

$$(28) \quad \tilde{w}_M(s, x) \geq \alpha' > 0 \quad \text{for } s \leq s_1 < 0$$

for some constants α' and s_1 and every large M .

Step 6: Further estimates on the time derivative of the limit solution

By the bound from above in Step 3, we obtain that

$$(29) \quad |w(t, x)| \leq C_1 + C_2(|t| + |x|),$$

where C_1 and C_2 are finite positive constants. Let now $(t_k, x_k) \in \{w > 0\}$ be a sequence such that

$$ct_k - x_k \cdot e \rightarrow -\infty.$$

Then by the result in Step 5,

$$(30) \quad d_k := \text{pardist}((t_k, x_k), \partial\{w > 0\}) \geq c_3 \sqrt{|t_k| + |x_k|^2}$$

for some constant $c_3 > 0$. So w is a solution of $\partial_t w - \Delta w = -v^0$ in $Q_{d_k}(t_k, x_k)$. Defining

$$z_k(t, x) := \frac{w(t_k + d_k^2 t, x_k + d_k x)}{d_k^2},$$

(29) and (30) imply that z_k is a solution of $\partial_t z_k - \Delta z_k = -v^0(x_k + d_k x)$ in $Q_1(0)$ satisfying

$$\sup_{Q_1(0)} |z_k| \leq C_4,$$

where C_4 is a constant not depending on k . Consequently $\partial_t w(t_k, x_k) = \partial_t z_k(0)$ is bounded, implying that $\limsup_{(t,x) \in \{w>0\}, ct-x \cdot e \rightarrow -\infty} |\partial_t w(t, x)| < +\infty$. Passing if necessary to a subsequence, we obtain by the periodicity of v^0 a limit z satisfying $\partial_t z - \Delta z = -\mu_0$ in $Q_1(0)$. Moreover we infer from the fact that \tilde{w} is periodic in the space variables that z is constant in the space variables. Thus $\partial_t z \equiv -\mu_0$ in $Q_1(0)$. From regularity theory of caloric functions it follows that

$$\lim_{(t,x) \in \{w>0\}, ct-x \cdot e \rightarrow -\infty} \partial_{tt} w = 0.$$

But then a combination of the comparison principle (22) and of (28) yield

$$-\mu_0 \leq \partial_t w \leq 0 \quad \text{on } \mathbf{R} \times \mathbf{R}^n$$

and

$$\partial_{tt} w \geq 0 \quad \text{on } \mathbf{R} \times \mathbf{R}^n.$$

This ends the proof of the Theorem 7.3.

8. NON-EXISTENCE OF PULSATING WAVES IN THE CASE OF CONSTANT INITIAL CONCENTRATION

We consider solutions u of the one-phase limit problem with constant initial concentration in *any finite dimension*, i.e.

$$(31) \quad \partial_t u - \partial_t \chi_{\{u \geq 0\}} = \Delta u \quad \text{in } \mathbf{R} \times \mathbf{R}^n,$$

and prove that u cannot be a non-trivial pulsating wave in the sense of (15), (16). More precisely:

Theorem 8.1. (Non-existence of pulsating waves for constant initial concentration)

Let u be a solution of (15), (16) in dimension $n \geq 1$ with $v^0 = \text{constant}$. Then $u(t, x) = u(t - e \cdot x/c, 0)$, i.e. u is a planar wave.

Proof of Theorem 8.1

We set $w(t, x) = -\int_t^{+\infty} u(s, x) ds \geq 0$. From the proof of Theorem 7.1 we know that $w > 0$ if and only if $u < 0$. As $\partial_t u \geq 0$ we obtain

$$\partial_t w = \Delta w - v^0 \chi_{\{w>0\}},$$

and w satisfies (17), (18). For any $\xi \in \mathbf{R}^n$, we define the “tangential difference”

$$z^\xi(t, x) = w\left(t - \frac{e \cdot \xi}{c}, x - \xi\right) - w(t, x)$$

which satisfies

$$(32) \quad (\partial_t - \Delta) z^\xi = -a z^\xi,$$

$$\text{where } 0 \leq a = \begin{cases} 0 & \text{if } z^\xi(t, x) = 0 \\ v^0 \left(\frac{\chi_{\{w(t-\frac{e \cdot \xi}{c}, x-\xi) > 0\}} - \chi_{\{w(t, x) > 0\}}}{w(t-\frac{e \cdot \xi}{c}, x-\xi) - w(t, x)} \right) & \text{if } z^\xi(t, x) \neq 0. \end{cases}$$

From (17), (18) and the definition of z^ξ we infer that

$$|\partial_t z^\xi| \leq 2\mu_0 = 2v^0 \text{ in } \mathbf{R}^{n+1},$$

$$\partial_t z^\xi(t, x) \longrightarrow 0 \text{ uniformly in } t, x, \xi \text{ as } ct - e \cdot x \longrightarrow -\infty.$$

Moreover (18) and (28) as well as the definition of z^ξ tell us that for some $s_0 \in (0, +\infty)$ not depending on ξ ,

$$(\partial_t - \Delta)z^\xi = 0 \quad \text{in} \quad \{|ct - e \cdot x| > s_0\}.$$

Furthermore we obtain from the comparison principle (see Lemmata 3.2 and 3.4 in [3]) that

$$(33) \quad |\partial_t z^\xi(t, x)| \leq 2v^0 e^{ct - e \cdot x + s_0}$$

and — integrating this estimate for $t \in (-\infty, \frac{s_0 + e \cdot x}{c})$ and using that $z^\xi = 0$ in $t \geq \frac{s_0 + e \cdot x}{c}$ — we obtain that z^ξ is bounded on \mathbf{R}^{n+1} by a constant not depending on ξ .

Liouville's theorem for the heat equation implies therefore that for each sequence (t_m, x_m) such that $ct_m - e \cdot x_m \rightarrow -\infty$, $z^\xi(t_m + \cdot, x_m + \cdot)$ converges locally uniformly in \mathbf{R}^{n+1} (and uniformly with respect to ξ) to a constant K depending on the choice of ξ and the sequence (t_m, x_m) . As we know that $\int_{x+[0,1]^n} z^\xi(t, y) dy = 0$ for every $(t, x) \in \mathbf{R}^{n+1}$ (see (18)), it follows that $K = 0$ and that

$$z^\xi(t + \cdot, x + \cdot) \rightarrow 0 \text{ locally uniformly in } \mathbf{R}^{n+1} \text{ as } ct - e \cdot x \rightarrow -\infty;$$

the convergence is also uniform with respect to ξ .

Finally we define

$$\eta(t, x) := \sup_{\xi \in \mathbf{R}^n} |z^\xi(t, x)|.$$

The function η is by (32) a bounded subcaloric function. Moreover, by construction,

$$\partial_y \eta(t - \frac{e \cdot y}{c}, x - y) \equiv 0.$$

But then $\eta(t, x) = f(ct - e \cdot x)$, $cf' - f'' \leq 0$ in \mathbf{R} , $f \in W_{\text{loc}}^{1,1}(\mathbf{R})$, $\lim_{s \rightarrow -\infty} f(s) = \lim_{s \rightarrow +\infty} f(s) = 0$ and f is bounded from above, implying that $f = 0$, that $\eta \equiv 0$ and that $w(t - \frac{e \cdot \xi}{c}, x - \xi) - w(t, x) = 0$ for every $t \in \mathbf{R}$ and $x, \xi \in \mathbf{R}^n$. We obtain the corresponding result for u .

9. CONCLUSIONS

In this section we try to take a conclusion of the previous sections.

Let us consider a sequence of solutions $(u_\varepsilon, v_\varepsilon)$ of the ε -problem (6) satisfying for example the assumptions in the time-increasing case of Theorem 4.1, and suppose

that they are getting closer and closer to one-phase pulsating waves as $\varepsilon \rightarrow 0$, i.e. for some $t_\varepsilon \rightarrow +\infty$,

$$(34) \quad \begin{cases} u_\varepsilon(t, x+k) = o(1) + u_\varepsilon(t - \frac{e \cdot k}{c}, x) & \text{for every } k \in \mathbf{Z}^n, (t, x) \in (t_\varepsilon, +\infty) \times \mathbf{R}^n \\ u_\varepsilon(t, x) = o(1) & \text{for } x \cdot e - ct \leq -t_\varepsilon \text{ and } \lim_{\varepsilon \rightarrow 0} \limsup_{x \cdot e - ct + t_\varepsilon \rightarrow +\infty} u_\varepsilon(t, x) = -\mu_0. \end{cases}$$

Translating t_ε to 0, we obtain by Theorem 4.1 a sequence $(u_\varepsilon(t_\varepsilon + t, x), v_\varepsilon(t_\varepsilon + t, x))$ that is locally compact in L^1 and u_ε converges to u satisfying all assumptions in Theorem 7.1 except the assumptions for $v^0(x)$. Let us assume $v^0 \geq c > 0$, i.e. there has been enough fuel everywhere initially, and let us show that v^0 is \mathbf{Z}^n -periodic: for the integrated function w as in section 7 and $k \in \mathbf{Z}^n$,

$$\begin{aligned} 0 &= (\partial_t - \Delta)(w(t, x+k) - w(t - \frac{e \cdot k}{c}, x)) \\ &= v^0(x) \chi_{\{w(t - \frac{e \cdot k}{c}, x) > 0\}} - v^0(x+k) \chi_{\{w(t, x+k) > 0\}} = (v^0(x) - v^0(x+k)) \chi_{\{w(t, x+k) > 0\}} \end{aligned}$$

and therefore, choosing $-t$ large, $v^0(x) - v^0(x+k) = 0$.

But then Theorem 8.1 tells us that u must either be one of the pulsating waves constructed in section 7, or a planar wave.

So our results when combined, strongly suggest a relation of the numerically observed pulsating waves for dimensions $n \geq 1$ and the pulsating waves constructed in section 7. Of course there remain possible alternatives, for examples our results do not exclude the non-existence of pulsating waves for small ε , or the strong convergence of pulsating waves to planar waves from far away, or the existence of non-trivial true two-phase pulsating waves in the case of constant v^0 . We leave these possibilities open to mathematical discussion. However in communication with mathematicians doing numerics for pulsating waves of the very systems mentioned in the introduction the above alternatives have been considered unlikely.

Let us also mention that the chaos reported in [1] for finite activation energy presents no contradiction to our results: we never claimed that the dynamics *around* the pulsating waves is simple.

On the other hand, our result – proving rigorously existence of pulsating waves for spatially inhomogeneous coefficients and non-existence for constant coefficients in any dimension – suggests that numerics – introducing *volens nolens* spatially inhomogeneous coefficients – makes the limit problem more unstable (see also the Appendix), and that grid information etc. *will* strongly influence the numerically obtained pulsating waves unless appropriate counter-measures are taken.

10. OPEN QUESTIONS

The most pressing task is of course to study for space dimension $n \geq 2$ the existence or non-existence of “peaking” of the solution in the negative phase. A related question is whether $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in L^∞ in the case of uniformly bounded initial data. Although this seems obvious, it is not clear how to prevent concentration close to the interface.

Another challenge is to use the information on the limit problem gained in the present paper to construct pulsating waves for the ε -problem.

Uniqueness for the limit problem (the irreversible Stefan problem for supercooled water) in general seems unlikely. One might however ask whether time-global

uniqueness holds in the case that u is strictly increasing in the x_1 -direction. By the result in [7] for the ill-posed Hele-Shaw problem, time-local uniqueness is likely to be true here, too.

11. APPENDIX: FORMAL STABILITY IN THE CASE OF ONE SPACE DIMENSION AND CONSTANT INITIAL CONCENTRATION

We consider solutions u of the one-phase limit problem with constant initial concentration in one space dimension, i.e.

$$(35) \quad \partial_t u - \partial_t \chi_{\{u \geq 0\}} = \partial_{xx} u \quad \text{in } \mathbf{R} \times \mathbf{R}$$

that are close to the traveling wave solution

$$(36) \quad \bar{u}(t, x) = -\max\left(1 - e^{-c(x-ct)}, 0\right)$$

moving with velocity $c > 0$.

Proposition 11.1. (Formal linear stability of one-dimensional traveling waves)

The traveling wave \bar{u} given by (36) is formally linearly stable with respect to equation (35).

Remark 11.2. In higher dimensions this result is no longer true. It is well known that a fingering instability occurs. However the pulsation phenomenon with which we are concerned in the present paper appears already in dimension 1.

Formal proof of Proposition 11.1:

Let us consider solutions u of (35) satisfying

$$(37) \quad \begin{cases} u(t, x) = 0 & \text{for } x \leq s(t) , \\ u(t, x) < 0 & \text{for } x > s(t) , \\ u(t, x) \longrightarrow -1 & \text{as } x - s(t) \rightarrow +\infty , \\ s'(t) = -\partial_x u(t, s(t) + 0^+) \geq 0 . \end{cases}$$

Let us remark that a simple analysis shows that we do not have a comparison principle for solutions of (37).

In order to analyze the stability we transform (37) by

$$v(t, y) := u(t, y + s(t)) ;$$

v satisfies $v(t, y) = 0$ for $y \leq 0$ and

$$(38) \quad \begin{cases} v(t, 0) = 0 , \\ v(t, y) < 0 & \text{for } y > 0 , \\ v(t, y) \longrightarrow -1 & \text{as } y \rightarrow +\infty , \\ \partial_t v = \partial_{yy} v + s'(t) \partial_y v & \text{on } \mathbf{R} \times (0, +\infty) , \\ s'(t) = -\partial_y v(t, 0^+) . \end{cases}$$

We now consider for $t > 0$ a perturbation of the traveling wave

$$\bar{v}(y) = -\max(1 - e^{-cy}, 0)$$

with velocity $c > 0$. In the formal expansion

$$\begin{cases} s(t) = ct + \varepsilon\gamma(t) + 0(\varepsilon^2) , \\ v = \bar{v} + \varepsilon w + 0(\varepsilon^2) , \end{cases}$$

the first order terms $w(t, y), \gamma(t)$ formally satisfy

$$\begin{cases} w(t, 0) = 0 , \\ w(t, y) \longrightarrow 0 \quad \text{as } y \rightarrow +\infty , \\ \partial_t w = \partial_{yy} w + c\partial_y w + \gamma'(t)\partial_y \bar{v} \quad \text{in } (0, +\infty) \times (0, +\infty) , \\ \gamma'(t) = -\partial_y w(t, 0^+) . \end{cases}$$

Let us look for solutions of the form

$$\begin{cases} w(t, y) = e^{\lambda t} W(y) , \\ \gamma'(t) = e^{\lambda t} , \end{cases}$$

where $\text{Re}(\lambda) \geq 0$. We obtain

$$W'' + cW' - \lambda W = ce^{-cy} ,$$

i.e.

$$W(y) = -\frac{c}{\lambda} e^{-cy} + \sum_{\pm} A_{\pm} e^{\mu_{\pm} y} ,$$

where

$$\mu_{\pm} = -c/2 \pm \sqrt{c^2/4 + \lambda} \quad \text{and} \quad \text{Re}(\sqrt{c^2/4 + \lambda}) > c/2 .$$

The function W can only be bounded if $A_+ = 0$ and, by $W(0) = 0$,

$$W(y) = -\frac{c}{\lambda} (e^{-cy} - e^{\mu_- y}) .$$

Finally the relation $\gamma'(t) = -\partial_y w(t, 0)$ implies

$$1 = \frac{c}{\lambda} (-c - \mu_-) .$$

The unique solution of this equation is $\lambda = 0$. Thus we formally proved stability of traveling waves.

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